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Symmetric Equilibrium in Pre-Auction Investment

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Abstract

We characterize the symmetric equilibria of a pre-auction game of investment in an internet ad auction environment. For a wide class of auction formats, we show a symmetric equilibrium exists for the pre-auction investment game, is essentially unique, and is the same for all auction formats in the class; we give sufficient conditions for the symmetric equilibrium to be in pure strategies, and to achieve the efficient level of investment.

1 Introduction

We investigate symmetric equilibria of a pre-auction game of investment in an internet ad auction environment, with multiple goods (advertisement positions) for sale. Bidders have one-dimensional types, whose distribution is determined by pre-auction investments. We focus on auction formats in which, if bidders are symmetric going into the auction, the goods that are sold are allocated efficiently (although supply may be inefficiently restricted by a reserve price). If pre-auction investment is unobservable to competitors, a broad class of auction formats yield the same set of symmetric equilibria in investment. We establish existence and essential uniqueness of the symmetric equilibrium; relate the efficiency of investment to the reserve price; give sufficient conditions for the symmetric equilibrium to be in pure strategies; and give sufficient conditions for it to achieve first-best investment.

Many of our results follow from a useful fact we have not seen reported elsewhere: with independent private values, pre-auction investment in bidders' own valuations is a potential game, with the potential function closely related to total surplus. Aside from facilitating proofs, this fact is of independent interest, as play in a potential game has nice convergence properties under many standard learning dynamics.

2 Related Literature

Hausch and Li (1991) establish that in a standard single-item auction setting with symmetric bidders, the first- and second-price auctions yield the same level of pre-auction investment when investment is covert, which we extend to more general auction formats and environments. Stegeman (1996) gives an analogous result when investment takes the form of costly participation. Arozamena and Cantillon (2004) contrast

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this equal-investment result with the case of *observable* investment, providing conditions under which the first-price auction leads to less (and less-than-efficient) value-enhancing (or cost-reducing) investment.

Rogerson (1992) observes that the Vickrey-Clarke-Groves mechanism provides efficient incentives for pre-auction investment. Given the computational burden of VCG in "large" auction settings, Hatfield, Kojima and Kominers (2018) and Akbarpour, Kominers, Li, and Milgrom (2023) study investment incentives in auctions which are approximations of VCG; depending on the details of the auction, these may or may not lead to approximately-efficient investment. Gershkov, Moldovanu, Strack and Zhang (2021) study revenue-maximizing multi-unit auctions when bidders make investment decisions after learning the realization of their private information.

Another form of pre-auction investment is endogenous information acquisition (see, for example, Bergemann and Valimaki (2002) and Persico (2000)). Li and Tian (2008) establish existence and uniqueness in special cases of a symmetric model with information acquisition. Kim and Koh (2022) and Pernoud and Gleyze (2023) consider various flexible forms of information acquisition; Bobkova (2024) examines the interaction between auction formats and the incentives to acquire information about common versus private value components.

The internet ad auction framework we use was introduced by Edelman, Ostrovsky and Schwarz (2007) and Varian (2009), and discussed further by Athey and Ellison (2011), Levin and Milgrom (2010), and Arnosti, Beck and Milgrom (2016), among others. This model nests the standard single-item auction setting as a special case.

3 Model

Our environment is the one considered by Edelman, Ostrovsky and Schwarz (2007), Varian (2009), and others. There are N items (ad positions adjacent to search results) to be allocated, ordered most valuable to least valuable, with "clickthrough rates" (the fraction of consumers shown the ad who will click to visit the corresponding website) $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_N$. There is a set $\mathcal{K}$ of bidders, with $K = |\mathcal{K}| > N$. For $k \in \mathcal{K}$, bidder k's value s_k (per arriving customer) is drawn from a distribution F_k and independent of $\{s_j\}_{j \neq k}$. The value to bidder k of being assigned position n is $\alpha_n s_k$, minus the payment made. We allow for different possible auction formats, whose equilibrium allocation and payment may depend on the distributions $\{F_k\}$ from which the various bidders' valuations are drawn.

We model pre-auction investment as follows. There is a set $\mathcal{F}$ of probability distributions with supports contained in an interval $[0, \overline{s}]$, and a cost function $c : \mathcal{F} \to \mathbb{R}_{\geq 0}$ assigning a cost to each, with $\min_{F \in \mathcal{F}} c(F) = 0$. Players simultaneously choose distributions $F_k \in \mathcal{F}$, incurring the corresponding costs $c(F_k)$. The auction then occurs, with each bidder's type s_k drawn (independently) from that bidder's chosen distribution F_k.

We assume investment is *covert*: bidders do not observe each others' choices of F_k, although they correctly infer them in equilibrium. (Thus, if a bidder plays a mixed strategy in equilibrium, at the time of the auction their opponents know their mixed strategy but not which distribution resulted.) We take as given equilibrium play in the auction itself, which may depend on the distribution the bidders' valuations are believed to come from; we focus on analyzing the game in which the bidders choose F_k. We refer to an equilibrium of this game as an *investment equilibrium*.

We associate each distribution $F \in \mathcal{F}$ with its CDF $F : [0, \overline{s}] \to [0, 1]$, and define a metric on $\mathcal{F}$ by defining the distance between two distributions as

$$d(F, F') \quad = \quad \inf \{\epsilon : |\{s : |F(s) - F'(s)| > \epsilon\}| \leq \epsilon\}$$

that is, F and F' are within ϵ of each other if their CDFs $F(s)$ and $F'(s)$ are within ϵ of each other except on a set of measure ϵ or less. We impose standard restrictions to ensure best-responses exist:

Assumption 1. *$\mathcal{F}$ is compact and c is continuous.*

Note that $\mathcal{F}$ can be either finite or infinite; Assumption 1 holds trivially if $\mathcal{F}$ is finite.

We consider auctions with a reserve price $r \geq 0$, such that the minimum "payment per click" is r (or the minimum payment to win slot n is $\alpha_n r$). Interpreting the reserve price as a constraint on the allocation, we say an allocation is *constrained efficient* if slots are allocated efficiently among the bidders whose valuations s_k exceed the reserve price. We say an auction format is *constrained efficient when symmetric* if whenever all bidders' valuations are believed to be drawn from the *same* distribution, equilibrium play in the auction leads to the constrained-efficient allocation. For example, the VCG mechanism, the equilibrium of the Generalized English Auction studied by Edelman, Ostrovsky and Schwarz (2007), a first-price ad auction, and an all-pay ad auction are all constrained efficient when symmetric; the first two give constrained-efficient allocations even when bidder valuations are drawn from different distributions.

How Our Model Relates to Others in the Literature

Lots of literature focuses on the special case where $\mathcal{F}$ is a family of parameterized distributions $\mathcal{F} = \{F_\eta\}_{\eta \in [0,1]}$, with $c(F_\eta) = C(\eta)$ increasing in η. When $F_{\eta'}$ first-order stochastically dominates F_η for $\eta' > \eta$, this is a classic model of value-enhancing investment. On the other hand, when F_η second-order stochastically dominates $F_{\eta'}$ for $\eta' > \eta$, we can interpret this as a model of costly information acquisition. (If each bidder receives a noisy signal x_k of their private value, the conditional expectation $E(s_k|x_k)$ serves as their private value in the auction, and a more precise signal x_k yields a mean-preserving spread in these interim expected values.) Our general model in Section 4 requires no specific structure beyond Assumption 1, and nests both of these as special cases; Section 5 studies the special case of one-dimensional value-enhancing investment.

Some papers consider a different model, where each bidder i draws a type $\theta_i \in \Theta$ from a fixed distribution F and chooses an investment level $a_i \in A$ at cost $C(a_i)$, resulting in a private value $v(\theta_i, a_i)$. When (as in Hausch and Li 1991) a_i is chosen before a bidder learns his type, our model nests this possibility by defining F_{a_i} as the distribution of $v(\theta_i, a_i)$, and $\mathcal{F}$ as $\{F_{a_i}\}_{a_i \in A}$. When a bidder learns his type θ_i before choosing his investment level (as in Gershkov et al. 2021), the mapping to our model is less obvious, but we can still nest such a model by considering a bidder making an ex ante choice of a *contingent* investment plan $\alpha_i : \Theta \to A$, defining a planned investment level $\alpha_i(\theta_i)$ for each possible realization of θ_i. For each such mapping, we can define F_{α_i} as the resulting distribution of private values $v(\theta_i, \alpha_i(\theta_i))$, with the corresponding (expected) investment cost $c(F_{\alpha_i}) = E_{\theta_i} C(\alpha_i(\theta_i))$.

4 Equilibrium Characterization

4.1 Symmetric Equilibria are the Same Across Auction Formats

Hausch and Li (1991) consider a standard single-item setting with independent private values and symmetric, one-dimensional pre-auction investment opportunities. They show that, subject to a second-order condition, there is a level of pre-auction investment that is a symmetric pure-strategy equilibrium for all "standard auctions." Here, we generalize this result somewhat: multiple items are allowed, the *entire set* of symmetric equilibria (pure or mixed) is the same for all auctions in our class, and we do not require a parametrization of $\mathcal{F}$ or a second-order condition. (We explore existence, and whether the equilibrium is in pure strategies, separately). Still, we see the main insight of identical pre-auction investment across auction formats as being Hausch and Li's, and a similar idea appears in Stegeman (1996); we give the result here to justify characterizing symmetric equilibria for just one auction format.

If bidder k is expected to play a mixed strategy σ_k over $\mathcal{F}$, his opponents' belief about the distribution of his valuation corresponds to the compound lottery

$$F_{\sigma_k}(s) \quad \equiv \quad \int_{\mathcal{F}} F(s) d\sigma_k(F)$$

(or the analogous sum if σ_k has finite support), which lies in the convex hull $\text{conv}(\mathcal{F})$ of $\mathcal{F}$. We can thus think of bidders' beliefs about each others' types as individual distributions in $\text{conv}(\mathcal{F})$, rather than as distributions over elements of $\mathcal{F}$. (We continue to think of bidders' mixed strategies, however, as distributions over $\mathcal{F}$ rather than points in $\text{conv}(\mathcal{F})$.)

For a particular auction format and a belief profile $\mathbf{F} = (F_{\sigma_1}, F_{\sigma_2}, \ldots, F_{\sigma_K}) \in (\text{conv}(\mathcal{F}))^K$, we define the auction's *equilibrium allocation rule at* $\mathbf{F}$ as the mapping from bidders' realized types to the allocation of prizes that occurs when (i) it's common knowledge that each bidder k's valuation follows the distribution F_{σ_k}, and (ii) bidders all play their equilibrium strategies in the auction given these beliefs. We begin by establishing that a bidder's incentives in the pre-auction investment game are determined by the auction's equilibrium allocation rule following the anticipated investment choices:

Lemma 1. *Fix a (not necessarily symmetric) profile of beliefs* $\mathbf{F} = (F_{\sigma_1}, F_{\sigma_2}, \ldots, F_{\sigma_K}) \in (\text{conv}(\mathcal{F}))^K$*. If two auctions implement the same equilibrium allocation rule at* $\mathbf{F}$*, then the private benefit to bidder* k *of covertly switching from* F_k *to* F'_k *is the same for the two auctions.*

Proof. Let $u_k(t_k, \mathbf{F})$ denote bidder k's equilibrium interim expected payoff when it's common knowledge that bidder valuations are drawn from $\mathbf{F}$ and bidder k's realized valuation is t_k. By usual envelope theorem arguments,

$$u_k(t_k, \mathbf{F}) \quad = \quad u_k(0, \mathbf{F}) + \int_0^{t_k} \sum_{n=1}^{N} \alpha_n \Pr(k \;\; wins \;\; prize \;\; n | s_k = s, \mathbf{F}) ds$$

where $\Pr(k \;\; wins \;\; prize \;\; n | s_k = s, \mathbf{F})$ is based on the equilibrium allocation rule at $\mathbf{F}$. If investment is covert, then whatever distribution bidder k chooses, once his valuation t_k is realized, this is his expected payoff: his opponents bid as if his value was drawn from F_{σ_k}, so he faces the same optimization problem as if he had drawn t_k from F_{σ_k}, so his optimal bid and expected payoff are the same. If he actually chooses F'_k, his ex ante expected payoff is

$$U_k(F'_k, \mathbf{F}) \quad = \quad \int_0^{\overline{s}} u_k(t_k, \mathbf{F}) dF'_k(t_k) - c(F'_k)$$

whether or not $F'_k = F_{\sigma_k}$. Similar to Myerson (1981), we can plug the expression above for $u_k(t_k, \mathbf{F})$ into this last expression, switch the order of integration, and evaluate the (new) integral to get

$$U_k(F'_k, \mathbf{F}) \quad = \quad u_k(0, \mathbf{F}) + \int_0^{\overline{s}} (1 - F'_k(s)) \sum_{n=1}^{N} \alpha_n \Pr(k \;\; wins \;\; prize \;\; n | s_k = s, \mathbf{F}) ds - c(F'_k)$$

and therefore

$$U_k(F'_k, \mathbf{F}) - U_k(F_k, \mathbf{F}) \quad = \quad \int_0^{\overline{s}} \sum_{n=1}^{N} \alpha_n (F_k(s) - F'_k(s)) \Pr(k \;\; wins \;\; prize \;\; n | s_k = s, \mathbf{F}) ds - c(F'_k) + c(F_k)$$

If two auctions implement the same equilibrium allocation at $\mathbf{F}$, then $\Pr(k \;\; wins \;\; prize \;\; n | s_k = s, \mathbf{F})$ is the same for both, so they have the same value for $U_k(F'_k, \mathbf{F}) - U_k(F_k, \mathbf{F})$, giving the result. □

Corollary 1. *If investment is covert and two auctions are both constrained efficient when symmetric, they have the same set of symmetric investment equilibria.*

Proof. Given a mixed strategy $\hat{\sigma}$ over $\mathcal{F}$, let $F_{\hat{\sigma}} \in \text{conv}(\mathcal{F})$ be the corresponding compound distribution, and $\hat{\mathbf{F}} = (F_{\hat{\sigma}}, F_{\hat{\sigma}}, \ldots, F_{\hat{\sigma}})$. All bidders playing $\hat{\sigma}$ is an equilibrium if and only if

$$U_k(F', \hat{\mathbf{F}}) - U_k(F, \hat{\mathbf{F}}) \quad \leq \quad 0 \qquad \text{for every } F \in \text{supp}(\hat{\sigma}) \text{ and } F' \in \mathcal{F} \tag{1}$$

Two auctions which are constrained efficient when symmetric implement the same allocation at $\hat{\mathbf{F}}$; by Lemma 1, $U_k(F', \hat{\mathbf{F}}) - U_k(F, \hat{\mathbf{F}})$ is the same for the two auctions, and (1) either holds for both or for neither, giving the result. □

To characterize the symmetric equilibria of all constrained-efficient-when-symmetric auctions, then, it suffices to characterize the symmetric equilibria of any one of them. We therefore focus on characterizing the symmetric investment equilibria of the Vickrey-Clarke-Groves auction with a reserve price.

4.2 In VCG, Pre-Auction Investment is a Potential Game

In a VCG auction with a possibly-binding reserve price $r \geq 0$, the equilibrium allocation rule does not depend on bidders' beliefs about the distributions chosen by their opponents; since bidders report their types truthfully, the n^{th} prize goes to the bidder with the n^{th} highest valuation, provided it exceeds the reserve. For ease of exposition, we focus on the case where the distributions in $\mathcal{F}$ have no point masses, and the probability of a tie is therefore zero. For $s \geq r$, then, if each opponent j's valuation is drawn from F_{σ_j},

$$\Pr(k \;\; wins \;\; prize \;\; n | s_k = s) \quad = \quad \sum_{A \subset \mathcal{K} - \{k\} : |A| = n-1} \left(\prod_{j \in A} (1 - F_{\sigma_j}(s)) \prod_{j \in \mathcal{K} - \{k\} - A} F_{\sigma_j}(s) \right)$$

where the sum is taken over the sets A that contain exactly $n-1$ of bidder k's opponents. (The summand is the probability that the $n-1$ opponents in the set A have valuations above s, while the other $K - n$ opponents have valuations below s; by summing over all subsets of k's opponents of size $n-1$, we get the probability that exactly $n-1$ opponents have valuations above s, and therefore the probability that bidder k has the n^{th} highest valuation and will win prize n.)

For a given profile of mixed strategies $\sigma = (\sigma_1, \ldots, \sigma_K)$, define

$$F_\sigma^{(1)}(s) = \prod_{k=1}^{K} F_{\sigma_k}(s)$$

as the CDF of the highest valuation among the K bidders, and more generally,

$$F_\sigma^{(n)}(s) = \sum_{i=1}^{n} \sum_{A \subset \mathcal{K}:|A|=i-1} \prod_{k \in A} (1 - F_{\sigma_k}(s)) \prod_{k \in \mathcal{K}-A} F_{\sigma_k}(s)$$

as the CDF of the n^{th} highest valuation. Extend the cost function c to mixed strategies in the natural way, defining

$$c(\sigma_k) = \int_{\mathcal{F}} c(F) d\sigma_k(F)$$

for $\sigma_k \in \Delta\mathcal{F}$; and define a function $P : (\Delta\mathcal{F})^K \to \mathbb{R}$ over the space of mixed strategy profiles by

$$P(\sigma_1, \ldots, \sigma_K) = -\sum_{n=1}^{N} \alpha_n \int_r^{\bar{s}} F_\sigma^{(n)}(s) ds - \sum_{k=1}^{K} c(\sigma_k)$$

Our next result is that $P(\cdot)$ is a potential function for the investment game preceding a VCG auction.

Lemma 2. *In VCG with reserve price r, for every $k \in K$ and profile σ_{-k} of bidder k's opponents' strategies,*

$$U_k(F'_k, \sigma_{-k}) - U_k(F_k, \sigma_{-k}) = P(F'_k, \sigma_{-k}) - P(F_k, \sigma_{-k}) \quad (2)$$

Thus, the pre-auction investment game is a potential game, with potential function P.

Proof. We will show that $U_k(F_k, \sigma_{-k}) - P(F_k, \sigma_{-k})$ does not depend on F_k, so that $U_k(F'_k, \sigma_{-k}) - P(F'_k, \sigma_{-k}) = U_k(F_k, \sigma_{-k}) - P(F_k, \sigma_{-k})$, which is equivalent to (2). Note that

$$\begin{aligned}
U_k(F_k, \sigma_{-k}) - P(F_k, \sigma_{-k}) &= \int_r^{\bar{s}} (1 - F_k(s)) \sum_{n=1}^{N} \alpha_n \Pr(k \ \ wins \ \ prize \ \ n | s_k = s) ds - c(F_k) \\
&\quad + \sum_{n=1}^{N} \alpha_n \int_r^{\bar{s}} F_\sigma^{(n)}(s) ds + \sum_{k'=1}^{K} c(\sigma_{k'}) \\
&= \int_r^{\bar{s}} \sum_{n=1}^{N} \alpha_n \Pr(k \ \ wins \ \ prize \ \ n | s_k = s) ds + \sum_{k' \neq k} c(\sigma_{k'}) \\
&\quad + \sum_{n=1}^{N} \alpha_n \int_r^{\bar{s}} \left[F_\sigma^{(n)}(s) - F_k(s) \Pr(k \ \ wins \ \ prize \ \ n | s_k = s) \right] ds
\end{aligned}$$

The first line does not depend on F_k, so it suffices to show that the expression in square brackets doesn't either.

Define $q_i(s)$ as the probability (given $\{F_{\sigma_{k'}}\}_{k' \neq k}$) that exactly i of bidder k's opponents have valuations above s. Then we can decompose $F_\sigma^{(n)}(s)$, the probability the n^{th}-highest valuation is below s, into two cases, based on whether $s_k > s$ or $s_k \leq s$. Summing the probabilities of those two cases,

$$\begin{aligned} F_\sigma^{(n)}(s) &= \Pr(s_k > s \text{ and at most } n-2 \text{ other bidders have values above } s) \\ &\quad + \Pr(s_k \leq s \text{ and at most } n-1 \text{ other bidders have values above } s) \\ &= (1 - F_k(s)) \sum_{i=0}^{n-2} q_i(s) + F_k(s) \sum_{i=0}^{n-1} q_i(s) \end{aligned}$$

Since k's probability of winning prize n when $s_k = s$ is $q_{n-1}(s)$, the term in the square brackets above, $F_\sigma^{(n)}(s) - F_k(s) \Pr(k \;\; wins \;\; prize \;\; n | s_k = s)$, simplifies to

$$(1 - F_k(s)) \sum_{i=0}^{n-2} q_i(s) + F_k(s) \sum_{i=0}^{n-1} q_i(s) - F_k(s) q_{n-1}(s) \quad = \quad \sum_{i=0}^{n-2} q_i(s)$$

Since this does not depend on F_k, $U_k(F_k, \sigma_{-k}) - P(F_k, \sigma_{-k})$ does not depend on F_k, proving the result. □

As noted in the introduction, investment being a potential game is of independent interest due to the convergence properties of learning in potential games. Swenson, Murray and Kar (2018), for example, show that for a broad class of potential games, best-response dynamics converge at an exponential rate to a pure-strategy Nash equilibrium; see section 13.6 of Sandholm (2015) for more.

4.3 Existence and Essential Uniqueness of Symmetric Equilibrium

Knowing that the investment game preceding VCG is a potential game helps us characterize symmetric equilibrium. We begin with an existence result, to ensure later results aren't vacuous.

Lemma 3. *In the covert investment game preceding VCG with reserve price r, a symmetric investment equilibrium exists.*

The proof, in the appendix, is by applying the appropriate fixed point theorem to the correspondence mapping a mixed strategy $\sigma \in \Delta\mathcal{F}$ to a bidder's best response set when the other $K-1$ bidders play σ.

Next, we establish essential uniqueness of the symmetric equilibrium, which takes several steps. We begin by showing that bidders' investment choices are strategic substitutes:

Lemma 4. *Pick two bidders j and k, and fix a profile $\sigma_{-j,k}$ of the other bidders' strategies.*

- *The private gain to bidder k of switching from σ_k to σ'_k is greater when $\sigma_j = \sigma_k$ than when $\sigma_j = \sigma'_k$.*
- *The difference is strict if $F_{\sigma'_k}$ and F_{σ_k} differ on a range that "matters," i.e., if there exists $s \geq r$ at which $F_{\sigma_k}(s) \neq F_{\sigma'_k}(s)$ and $|\{k' \notin \{j,k\} : F_{\sigma_{k'}}(s) = 0\}| < N$.*

Proof. The private gain from switching from σ_k to σ'_k is

$$U_k(\sigma'_k, \cdot) - U_k(\sigma_k, \cdot) \quad = \quad \int_0^{\bar{s}} \sum_{n=1}^{N} \alpha_n (F_{\sigma_k}(s) - F_{\sigma'_k}(s)) \Pr(k \;\; wins \;\; prize \;\; n | s_k = s) ds - c(\sigma'_k) + c(\sigma_k)$$

Letting $z_i(s)$ be the probability (given $\sigma_{-j,k}$) that exactly i bidders other than j and k have valuations above s, then for $s \geq r$,

$$\Pr(k \;\; wins \;\; prize \;\; n | s_k = s) \quad = \quad F_{\sigma_j}(s) z_{n-1}(s) + (1 - F_{\sigma_j}(s)) z_{n-2}(s)$$

for $n > 1$, and $F_{\sigma_j}(s) z_0(s)$ for $n = 1$. Defining $z_{-1} = 0$, we can then write

$$U_k(\sigma'_k, \cdot) - U_k(\sigma_k, \cdot) \quad = \quad \int_r^{\bar{s}} (F_{\sigma_k}(s) - F_{\sigma'_k}(s)) \sum_{n=1}^{N} \alpha_n \left[F_{\sigma_j}(s) z_{n-1}(s) + (1 - F_{\sigma_j}(s)) z_{n-2}(s) \right] ds - c(\sigma'_k) + c(\sigma_k)$$

Some algebra (shown in the supplemental appendix) lets us rearrange this to

$$\begin{aligned} U_k(\sigma'_k, \cdot) - U_k(\sigma_k, \cdot) \quad &= \quad \int_r^{\bar{s}} (F_{\sigma_k}(s) - F_{\sigma'_k}(s)) F_{\sigma_j}(s) \left[\alpha_N z_{N-1}(s) + \sum_{n=1}^{N} (\alpha_n - \alpha_{n+1}) z_{n-1}(s) \right] ds \\ &\quad + \int_r^{\bar{s}} (F_{\sigma_k}(s) - F_{\sigma'_k}(s)) \sum_{n=1}^{N} \alpha_n z_{n-2}(s) ds - c(\sigma'_k) + c(\sigma_k) \end{aligned}$$

The second line does not depend on F_{σ_j}. As for the first, since α_n is decreasing in n, the term in square brackets is positive, and therefore the integrand in the first line is increasing in $F_{\sigma_j}(s)$ when $F_{\sigma_k}(s) > F_{\sigma'_k}(s)$, and decreasing in $F_{\sigma_j}(s)$ when $F_{\sigma_k}(s) < F_{\sigma'_k}(s)$. For each s, then, the integrand is greater when $F_{\sigma_j} = F_{\sigma_k}$ than when $F_{\sigma_j} = F_{\sigma'_k}$, so $U_k(\sigma'_k) - U_k(\sigma_k)$ is greater when $F_{\sigma_j} = F_{\sigma_k}$. Finally, the difference is strict if $F_{\sigma_k}(s) - F_{\sigma'_k}(s) \neq 0$ on a range where $\sum_{n-1}^{N} z_{n-1}(s) \neq 0$, i.e., if there is positive probability that fewer than N other bidders have valuations above s. $\square$

Next, define

$$\tilde{P}(\sigma) \quad = \quad P(\sigma, \sigma, \ldots, \sigma)$$

as the potential function evaluated at a symmetric strategy profile.

Lemma 5. *If a symmetric equilibrium exists where all players play $\hat{\sigma}$, then $\hat{\sigma} \in \arg\max_{\sigma \in \Delta \mathcal{F}} \tilde{P}(\sigma)$.*

Proof. Suppose not, so $\hat{\sigma}$ is a symmetric equilibrium but $\tilde{P}(\sigma') > \tilde{P}(\hat{\sigma})$ for some σ'. Then all bidders switching from $\hat{\sigma}$ to σ' increases P. By Lemma 4, bidder k gains strictly more by switching from $\hat{\sigma}$ to σ' when more of his opponents are playing $\hat{\sigma}$; by Lemma 2, when bidder k changes strategies, the change in k's payoff matches the change in P. Together, these imply that if all bidders are playing $\hat{\sigma}$, switching a single bidder to σ' strictly increases that bidder's expected payoff, contradicting $\hat{\sigma}$ being an equilibrium. $\square$

Lemma 6. *If $\sigma', \sigma'' \in \arg\max_{\sigma \in \Delta \mathcal{F}} \tilde{P}(\sigma)$, then $F_{\sigma'}(s) = F_{\sigma''}(s)$ for all $s \in (r, \bar{s})$.*

Proof. First, note that

$$\tilde{P}(\sigma) \quad = \quad -\sum_{n=1}^{N} \alpha_n \int_r^{\bar{s}} \left[\sum_{i=1}^{n} \binom{K}{i-1} (1 - F_\sigma(s))^{i-1} (F_\sigma(s))^{K-i+1} \right] ds - K c(\sigma)$$

We can write this as

$$\tilde{P}(\sigma) \quad = \quad -\int_r^{\bar{s}} \phi(F_\sigma(s)) ds - K c(\sigma)$$

where

$$\phi(x) \quad \equiv \quad \sum_{n=1}^{N} \alpha_n \sum_{i=1}^{n} \binom{K}{i-1} (1-x)^{i-1} x^{K-i+1}$$

This new function ϕ is strictly convex,[1] so $\phi\left(\frac{1}{2}F_{\sigma'}(s)+\frac{1}{2}F_{\sigma''}(s)\right) < \frac{1}{2}\phi(F_{\sigma'}(s))+\frac{1}{2}\phi(F_{\sigma''}(s))$ if $F_{\sigma'}(s) \neq F_{\sigma''}(s)$. Further, since CDFs are right-continuous, if $F_{\sigma'}(s) \neq F_{\sigma''}(s)$ at some s, they must be unequal on an open neighborhood to the right of s. Combined with the fact that $c\left(\frac{1}{2}\sigma'+\frac{1}{2}\sigma''\right) = \frac{1}{2}c(\sigma')+\frac{1}{2}c(\sigma'')$, this implies that if $F_{\sigma'}(s) \neq F_{\sigma''}(s)$ anywhere on $(r, \overline{s})$, then $\tilde{P}\left(\frac{1}{2}\sigma'+\frac{1}{2}\sigma''\right) > \frac{1}{2}\tilde{P}(\sigma')+\frac{1}{2}\tilde{P}(\sigma'')$, contradicting σ' and σ'' both maximizing $\tilde{P}$. □

Lemma 6 implies that all maximizers of $\tilde{P}$ give the same distribution of valuations F_σ above r; to give the same value of $\tilde{P}$, they must also have the same cost $c(\sigma)$. Thus, $\tilde{P}$ has an effectively unique maximizer; Lemma 5 says this maximizer is the unique candidate for a symmetric equilibrium. Combining these with the existence result (Lemma 3) and Corollary 1 gives the following:

Theorem 1. *Suppose Assumption 1 holds and investment is covert. Consider any auction format which is constrained efficient when symmetric.*

1. *A symmetric investment equilibrium exists (possibly in mixed strategies)*
2. *The symmetric investment equilibrium is essentially unique, in that all symmetric equilibria have the same expected cost per bidder and the same distribution of bidder valuations above r*
3. *The symmetric investment equilibrium is the same for all constrained-efficient-when-symmetric auction formats with the same reserve price*

4.4 Efficiency of Equilibrium Investment

Next, we consider total surplus, which is the expected surplus of the bidders plus the expected profit of the seller. Let $v_0 \geq 0$ denote the seller's cost per click, so that $\alpha_n v_0$ is the cost to the seller of assigning slot n, or the value the seller earns from not assigning it.

The VCG mechanism with $r = v_0$ gives the ex post efficient allocation. As a result, first-best would be achieved by VCG with $r = v_0$, preceded by any investment profile that maximizes total surplus given that mechanism. With $r \neq v_0$, no level of ex ante investment can match this level of total surplus, but we can define two benchmarks:

- Given r, *constrained-efficient investment* is any investment profile (symmetric or asymmetric) that maximizes expected total surplus given that reserve price. This coincides with first-best investment when $r = v_0$.
- Given r, *second best investment* is the *symmetric* investment profile that maximizes expected total surplus among symmetric profiles given that reserve price. This coincides with constrained-efficient investment when the latter happens to be symmetric.

Total surplus at a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_K)$ is

$$W(\sigma) \quad = \quad \sum_{n=1}^{N} \alpha_n \int_r^{\overline{s}} (s - v_0) dF_\sigma^{(n)}(s) - \sum_{k=1}^{K} c(\sigma_k)$$

[1]Differentiating and then simplifying gives $\phi'(x) = K\sum_{n=1}^{N} \alpha_n \binom{K-1}{n-1}(1-x)^{n-1}x^{K-n}$ and

$$\phi''(x) \quad = \quad K(K-1)\textstyle\sum_{n=1}^{N-1}(\alpha_n - \alpha_{n+1})\binom{K-2}{n-1}(1-x)^{n-1}x^{K-n-1} + K(K-1)\alpha_N\binom{K-2}{N-1}(1-x)^{N-1}x^{K-N-1}$$

which is strictly positive since a_n is decreasing; algebra is shown in the supplemental appendix.

Integrating by parts and rearranging gives

$$W(\sigma) \quad = \quad P(\sigma) + (\overline{s} - v_0)\sum_{n=1}^{N} \alpha_n - (r - v_0)\sum_{n=1}^{N} \alpha_n F_\sigma^{(n)}(r)$$

This leads to the following:

Theorem 2. *Fix a constrained-efficient-when-symmetric auction with reserve price r. Let $\tilde{\sigma}$ be the symmetric equilibrium; and let $(\sigma^*, \sigma^*, \ldots, \sigma^*)$ be any second-best investment profile given r.*

- *If $r = v_0$ or $r = 0$, $F_{\tilde{\sigma}}(r) = F_{\sigma^*}(r)$*
- *$F_{\tilde{\sigma}}(r) \geq F_{\sigma^*}(r)$ if $r > v_0$, and $F_{\tilde{\sigma}}(r) \leq F_{\sigma^*}(r)$ if $r < v_0$*

Proof. For the first part, if $r = v_0$, then $(r - v_0)\sum_n \alpha_n F_\sigma^{(n)}(r)$ vanishes; likewise if $r = 0$, since then $F_\sigma^{(n)}(r) = 0$ for all σ. In either case, up to an additive constant, $W = P$; since a symmetric equilibrium must be a symmetric maximizer of P, it's therefore a symmetric maximizer of W.

For the second part, suppose $r > v_0$ and $F_{\tilde{\sigma}}(r) < F_{\sigma^*}(r)$. This implies $F_{\tilde{\sigma}}^{(n)}(r) < F_{\sigma^*}^{(n)}(r)$ for each n. Since $\tilde{\sigma}$ is the symmetric maximizer of P, $P(\tilde{\sigma}, \tilde{\sigma}, \ldots, \tilde{\sigma}) \geq P(\sigma^*, \sigma^*, \ldots, \sigma^*)$. Together, these would imply $W(\tilde{\sigma}, \tilde{\sigma}, \ldots, \tilde{\sigma}) > W(\sigma^*, \sigma^*, \ldots, \sigma^*)$, contradicting σ^* being the symmetric maximizer of W. An analogous contradiction occurs if $r < v_0$ and $F_{\tilde{\sigma}}(r) > F_{\sigma^*}(r)$. □

Theorem 2 says that when $r \neq v_0$, in addition to distorting the allocation, the reserve price can distort equilibrium investment away from the efficient symmetric level given r. For example, a reserve above the seller's cost leads to investment levels at which the reserve price binds more often than it would at efficient investment. Whether this is a sign of "overinvestment" or "underinvestment" depends on the details of $\mathcal{F}$ and $c(\cdot)$; Corollary 3 in the next section considers a special case where the interpretation is more clear.

When $r = v_0$, the symmetric investment equilibrium coincides with the symmetric maximizer of total surplus. This may or may not, however, be the global maximizer of total surplus, i.e., the efficient investment profile when we consider asymmetric profiles as well. Define an equilibrium to be in *meaningfully mixed strategies* if it involves bidders mixing between two or more strategies F and F' such that $F(s) \neq F'(s)$ for some $s > r$.

Theorem 3. *If the symmetric equilibrium is in meaningfully mixed strategies, it does not achieve constrained-efficient investment.*

Proof. For simplicity, we focus on the case where $r = v_0$, so that P and W coincide up to a constant. Consider first a mixed strategy over two pure strategies, $\sigma = pF' + (1-p)F''$. Since it's an equilibrium, player 1 is indifferent among F', F'', and the mixture σ, meaning $P(\sigma, \sigma) = P(F', \sigma)$. But with player 1 mixing between F' and F'', player 2 was indifferent between F' and F''. If player 1 plays F', then player 2 strictly prefers F'' to F' (Lemma 4). So starting from any strictly mixed equilibrium, P strictly increases if we switch player 1 to one of the two pure strategies and player 2 to the other.

For mixtures over more than two strategies, simply think of the strategy as a mixture over two distinct mixed strategies, and the same argument holds. The proof for $r \neq v_0$ is in the appendix. □

Note that the inverse of Theorem 3 does not hold: a symmetric pure-strategy equilibrium still need not be constrained-efficient. For example, let $K = 2$ (two bidders), $N = 1$ (one prize), and $r = v_0 = 0$. Let

$\mathcal{F}$ be the family $\{F_\eta\}_{\eta\in[0,1]}$ of discrete distributions each putting weight η on valuation 1 and weight $1-\eta$ on valuation 0; and let $c(F_\eta) = \frac{1}{4}\eta + \frac{1}{4}\eta^2$. In the supplemental appendix, we calculate that the symmetric equilibrium is for both bidders to play the pure strategy $F_{1/2}$; but this gives strictly lower expected surplus than one bidder choosing F_0 and the other choosing F_1.

Theorem 3 says constrained-efficient investment is only possible when the symmetric equilibrium is in pure strategies, so we'd like to know when this occurs, and under what additional conditions constrained-efficient investment follows. To say more, we'll consider the common special case where $\mathcal{F}$ is a family of distributions parameterized by a one-dimensional parameter.

5 Further Results for a Parameterized Model

Let $\mathcal{F}$ be a family of distributions $\{F_\eta\}_{\eta\in[0,1]}$, parameterized by a continuous one-dimensional parameter η, with $c(F_\eta) = C(\eta)$. Assume $C(\cdot)$ is strictly increasing, and $F_{\eta'}$ first-order stochastically dominates F_η if $\eta' > \eta$. Further, assume $F_\eta(s)$ is continuous and twice differentiable in η for each s, and $C(\cdot)$ is twice differentiable as well. Finally, assume neither $\eta_1 = \eta_2 = \cdots = \eta_K = 0$ nor $\eta_1 = \eta_2 = \cdots = \eta_K = 1$ is an equilibrium.

We do not constrain $C(\cdot)$ to be convex. In fact, since we have not specified how $F_\eta(\cdot)$ depends on η, the "scale" of η is arbitrary, and the choice of $C(\cdot)$ can be thought of effectively as a normalization; our results below depend only on the properties of $F_{C^{-1}(z)}(\cdot)$.

Consider first a single bidder's problem when the other bidders are all playing the same, possibly mixed strategy σ. Note that

$$\Pr(k \;\; wins \;\; prize \;\; n | s_i = s) \quad = \quad \binom{K-1}{n-1} (1-F_\sigma(s))^{n-1} (F_\sigma(s))^{K-n}$$

so if we think of a bidder choosing η from $[0,1]$ rather than F_η from $\mathcal{F}$, the bidder's expected payoff is

$$U_k(\eta) \quad = \quad \int_r^{\overline{s}} (1-F_\eta(s)) \sum_{n=1}^N \alpha_n \binom{K-1}{n-1} (1-F_\sigma(s))^{n-1} (F_\sigma(s))^{K-n} ds - C(\eta) \tag{3}$$

It's helpful to reframe bidder k's problem as choosing how much to invest, $z = C(\eta)$, rather than choosing η; in that case, then, the expected payoff, and its derivatives, are

$$U_k(z) \quad = \quad \int_r^{\overline{s}} (1-F_{C^{-1}(z)}(s)) \sum_{n=1}^N \alpha_n \binom{K-1}{n-1} (1-F_\sigma(s))^{n-1} (F_\sigma(s))^{K-n} ds - z$$

$$U_k'(z) \quad = \quad -\int_r^{\overline{s}} \frac{\partial F_{C^{-1}(z)}(s)}{\partial z} \sum_{n=1}^N \alpha_n \binom{K-1}{n-1} (1-F_\sigma(s))^{n-1} (F_\sigma(s))^{K-n} ds - 1$$

$$U_k''(z) \quad = \quad -\int_r^{\overline{s}} \frac{\partial^2 F_{C^{-1}(z)}(s)}{\partial z^2} \sum_{n=1}^N \alpha_n \binom{K-1}{n-1} (1-F_\sigma(s))^{n-1} (F_\sigma(s))^{K-n} ds$$

This makes it transparent that a bidder's problem is concave if $F_{C^{-1}(z)}(s)$ is convex (in z) for every s, or $1-F_{C^{-1}(z)}(s)$ concave. The latter has a natural interpretation as "decreasing returns," as it means that for any s, the likelihood of receiving a valuation above s is concave in the amount of money the bidder invests.

Assumption 2. $\frac{\partial^2 F_{C^{-1}(z)}(s)}{\partial z^2} \geq 0$ *for all s, with strict inequality for s in some open interval $(\bar{s}-\epsilon, \bar{s})$.*

This ensures strict concavity of the bidder's problem, by ensuring that $\frac{\partial^2 F_{C^{-1}(z)}(s)}{\partial z^2}(F_\sigma(s))^{K-n}$ is strictly positive when integrated over $[r, \bar{s}]$. Note that

$$\frac{\partial^2 F_{C^{-1}(z)}(s)}{\partial z^2} \quad = \quad \frac{\partial^2 F_\eta(s)}{\partial \eta^2} - \frac{\partial F_\eta(s)}{\partial \eta}\frac{C''(\eta)}{C'(\eta)}$$

at $\eta = C^{-1}(z)$, and $\frac{\partial F_\eta(s)}{\partial \eta} \leq 0$, so Assumption 2 holds if F_η and C are both convex in η.[2]

Theorem 4. *Under Assumption 2, the unique symmetric investment equilibrium is in pure strategies.*

Proof. Under Assumption 2, a bidder's problem is strictly concave in η, so the first-order condition indicates a best-response.

Consider the problem of maximizing $\tilde{P}$ over pure strategies η,

$$\max_\eta \left\{ -\sum_{n=1}^N \alpha_n \int_r^{\bar{s}} \left[\sum_{i=1}^n \binom{K}{i-1} (1-F_\eta(s))^{i-1}(F_\eta(s))^{K-i+1} \right] ds - KC(\eta) \right\}$$

Since F_η is continuous in η, $\tilde{P}(F_\eta)$ is continuous in η, and since $\eta \in [0,1]$, a maximizer must exist. Assuming an interior maximum,[3] the first-order condition simplifies to (algebra in supplemental appendix)

$$\tilde{P}'(\eta) \quad = \quad -K \int_r^{\bar{s}} \sum_{n=1}^N \alpha_n \left[\binom{K-1}{n-1} (1-F_\eta(s))^{n-1}(F_\eta(s))^{K-n} \frac{\partial F_\eta(s)}{\partial \eta} \right] ds - KC'(\eta)$$

On the other hand, differentiating equation (3) above gives

$$U_k'(\eta) \quad = \quad -\int_r^{\bar{s}} \frac{\partial F_\eta(s)}{\partial \eta} \sum_{n=1}^N \alpha_n \binom{K-1}{n-1} (1-F_\sigma(s))^{n-1}(F_\sigma(s))^{K-n} ds - C'(\eta)$$

when a bidder's opponents are all playing the strategy σ. This means $\tilde{P}'(\eta) = 0$ is identical to $U_k'(\eta) = 0$ at $\sigma_{-k} = \eta$, so for $\eta^* = \arg\max_\eta \tilde{P}(F_\eta)$, F_{η^*} is a best-response to $\sigma_{-k} = \eta^*$ and therefore forms a symmetric equilibrium. □

Corollary 2. *Under Assumption 2, the symmetric equilibrium investment level decreases in r, decreases in K, and decreases as $C(\cdot)$ scales up uniformly, i.e., decreases in κ if $C(\cdot)$ is of the form $C(\eta) = \kappa \underline{C}(\eta)$ for some increasing function $\underline{C}(\cdot)$.*

Proof. Given Lemma 5, the symmetric equilibrium η^* maximizes $\tilde{P}(\eta)$, so it also maximizes $\frac{1}{K}\tilde{P}(\eta)$;

$$\frac{\partial}{\partial \eta}\left(\frac{1}{K}\tilde{P}(\eta)\right) \quad = \quad \int_r^{\bar{s}} \left(-\frac{\partial F_\eta(s)}{\partial \eta}\right) \sum_{n=1}^N \alpha_n \left[\binom{K-1}{n-1} (1-F_\eta(s))^{n-1}(F_\eta(s))^{K-n} \right] ds - C'(\eta)$$

[2] To keep signs straight, we can also think of this as $\frac{\partial^2 F_{C^{-1}(z)}(s)}{\partial z^2} = -\frac{\partial^2(1-F_\eta(s))}{\partial \eta^2} + \frac{\partial(1-F_\eta(s))}{\partial \eta}\frac{C''(\eta)}{C'(\eta)}$, where now $1-F_\eta(s)$ and C are both increasing in η. Assumption 2 is therefore equivalent to $\frac{-\partial^2(1-F_\eta(s))/\partial \eta^2}{\partial(1-F_\eta(s))/\partial \eta} > \frac{-C''(\eta)}{C'(\eta)}$, that is, to $1-F_\eta$ being "more concave" in η than C is, in the sense of having a higher, or less negative, Arrow-Pratt coefficient.

[3] If $\tilde{P}$ were maximized at $\eta = 0$, then $\tilde{P}'(0) \leq 0$. By the next step above, this would imply $\frac{\partial U_k}{\partial \eta_k} \leq 0$ at $\eta_1 = \eta_2 = \ldots = \eta_K = 0$. By concavity of U_k (which follows from Assumption 2), this would imply $\eta_1 = \eta_2 = \ldots = \eta_K = 0$ was an equilibrium, violating one of our starting assumptions. The analogous argument holds if $\tilde{P}$ were maximized at $\eta = 1$.

Since $\frac{\partial F_\eta(s)}{\partial \eta} < 0$, the integrand is positive, so the integral (hence the whole expression) is decreasing in r. If $C(\eta) = \kappa \underline{C}(\eta)$, then this is also decreasing in κ. To see it's decreasing in K, note that with some algebra (shown in the supplemental appendix),

$$\sum_{n=1}^{N} \alpha_n \binom{K-1}{n-1} (1 - F_\eta(s))^{n-1} (F_\eta(s))^{K-n} \quad = \quad \alpha_N F_\eta^{(N:K-1)}(s) + \sum_{n=1}^{N-1} (\alpha_n - \alpha_{n+1}) F_\eta^{(n:K-1)}(s)$$

where $F_\eta^{(n:K-1)}(s)$ is the probability that, out of $K-1$ independent draws from the distribution F_η, the n^{th}-highest draw is below s. Since α_n is decreasing, the "coefficients" $(\alpha_n - \alpha_{n+1})$ are all positive; and $F_\eta^{(n:K-1)}(s)$ is decreasing in K (the more bidders, the more likely at least $n+1$ have valuations above s), so the whole expression is decreasing in K. Thus, $\frac{1}{K}\tilde{P}$ has increasing differences in η and $-r$, $-\kappa$, and $-K$, so its unique maximizer (the symmetric equilibrium) is decreasing in all of these. □

Corollary 3. *Suppose Assumption 2 holds. Let $\tilde{\eta}$ denote the symmetric investment equilibrium, and let η^* be the second-best investment level (the level of symmetric investment that maximizes total surplus given r).*

- *if $r = v_0$ or $r = 0$, $\tilde{\eta} = \eta^*$*
- *if $r > v_0$, $\tilde{\eta} \leq \eta^*$*
- *if $r < v_0$, $\tilde{\eta} \geq \eta^*$*

Proof. As noted in the proof of Theorem 2, if $r = v_0$ or $r = 0$, $\tilde{P} = \tilde{W}$ up to a constant, so the symmetric surplus maximizer is the symmetric equilibrium. The last two parts follow directly from Theorem 2. □

Corollary 3 says that when $r = v_0$, the symmetric equilibrium investment level is the symmetric investment level that maximizes total surplus. However, as noted above, this need not be the investment profile that maximizes surplus overall, since surplus may be higher at an asymmetric investment profile. Following Theorem 3, we offered an example where the pure-strategy symmetric equilibrium gave strictly lower surplus than an asymmetric investment profile. In that example, if C were more convex – say, $C(\eta) = \frac{1}{2}\eta^2$ instead of $\frac{1}{4}\eta + \frac{1}{4}\eta^2$ – then the symmetric equilibrium would be first best, as $W(\eta_1, \eta_2)$ (total surplus as a function of both bidders' investment choices) would be strictly concave. This generalizes, and we can show that fixing any parameterized set of distributions $\{F_\eta\}$, if C is sufficiently convex, the symmetric equilibrium at $r = v_0$ achieves first-best.

Theorem 5. *Fix $\{F_\eta\}$. Suppose that (i) Assumption 2 holds, and (ii) there is some $M < \infty$ such that for all η and s, $\|\frac{\partial F_\eta(s)}{\partial \eta}\| \leq M$ and $\|\frac{\partial^2 F_\eta(s)}{\partial \eta^2}\| \leq M$.*

Then there exists $E < \infty$ such that if $C''(\eta) > E$ for all η, then at $r = v_0$, the symmetric investment equilibrium achieves first-best investment, and is therefore first-best.

Proof. Define $W(\eta_1, \ldots, \eta_K)$ as expected total surplus given pre-auction investment levels. The key step of the proof, done in the appendix, is to show that for C sufficiently convex, W is strictly concave. By Theorem 4, under Assumption 2, the symmetric equilibrium is the pure strategy $\tilde{\eta}$ that maximizes $\tilde{P}(F_\eta)$. Since it's an equilibrium, $\tilde{\eta}$ solves $\max_\eta P(F_\eta, F_{\tilde{\eta}}, \ldots, F_{\tilde{\eta}})$, and $\tilde{\eta}$ therefore satisfies the K first-order conditions for maximizing P at $\sigma_{-k} = \tilde{\eta}$. With $r = v_0$, as noted earlier, $W(\cdot) = P(\cdot)$ up to an additive constant, and so $(\tilde{\eta}, \ldots, \tilde{\eta})$ likewise satisfies the K first-order conditions for maximizing W; since W is strictly concave, this implies the symmetric equilibrium is the global maximizer of W. □

6 Conclusion

We've established that for any auction format implementing the constrained-efficient allocation when bidders make identical investment choices, a symmetric equilibrium in the covert pre-auction investment game exists, is essentially unique, and is the same across all such auction formats. Further, when this equilibrium is in mixed strategies, the equilibrium investment profile is inefficient; but in a parameterized setting when the reserve is set efficiently and the cost function sufficiently convex, the equilibrium is in pure strategies and achieves first-best investment.

One interpretation of these results is that when pre-auction investment is covert, revenue equivalence extends to auctions with endogenous valuations. Of course, this depends heavily on both a symmetric environment and a focus on symmetric equilibrium. When different auction formats yield different equilibrium allocations when valuations are asymmetric, they will have different asymmetric equilibria for the pre-auction investment game; in cases where the efficient investment profile is asymmetric, the choice of auction format may therefore be important.

Appendix – Omitted Proofs

A.1 Proof of Lemma 3 (existence of symmetric equilibrium)

The result is a direct application of Glicksberg's (1952) or Fan's (1952) extensions of Kakutani's fixed point theorem to infinite-dimensional strategy spaces. Glicksberg (1952) proves existence of a Nash equilibrium in a two-player game with compact Hausdorff pure strategy spaces A_1 and A_2 by applying his fixed point theorem to the correspondence from $\Delta A_1 \times \Delta A_2$ to itself that takes a mixed strategy profile (σ_1, σ_2) to the set $(BR_1(\sigma_2), BR_2(\sigma_1))$, whose fixed point is a mixed strategy equilibrium. To prove a *symmetric* equilibrium exists for a game where the players all have pure strategy space $\mathcal{F}$, we make the same argument, but for the correspondence from $\Delta\mathcal{F}$ to $\Delta\mathcal{F}$ taking a mixed strategy σ to the set of one player's best responses when the other $K-1$ players are playing σ.[4] The only requirements for this to work are that the pure strategy space $\mathcal{F}$ be compact (assumed) and Hausdorff (trivial since we placed a metric on it), and that the payoff function is continuous as a function of the pure strategies.[5]

[4]For good intuition on the underlying mechanics of the proof, see also lecture 6 of Asu Ozdaglar's 2010 lecture notes for the MIT course "Game Theory with Engineering Applications," available at https://ocw.mit.edu/courses/6-254-game-theory-with-engineering-applications-spring-2010/pages/lecture-notes/

[5]To see the latter, note that for the VCG mechanism,

$$U_k(F_k, F_{-k}) \quad = \quad \sum_{n=1}^{N} \alpha_n \int_0^{\bar{s}} (1 - F_k(s)) \Pr(n-1 \text{ of bidder } k\text{'s opponents have valuations above } s) ds - c(F_k)$$

This is continuous in F_k, since c is continuous, and a change from F_k to F'_k with $\|F_k - F'_k\| \leq \epsilon$ changes $1 - F_k(s)$ by less than ϵ except on a set of measure ϵ, and changes it by less than 1 everywhere, so the overall change in the n^{th} integral is no more than $2\epsilon\alpha_n$. Likewise, it's continuous in F_{-k} because an ϵ change in one of the other bidders' value distributions likewise changes each of the probabilities by less than ϵ except on a set of s having measure less than ϵ.

A.2 Proof of Theorem 3 when $r \neq v_0$

In the text, we proved Theorem 3 for $r = v_0$. When $r \neq v_0$, that $P(\cdot)$ and $W(\cdot)$ differ by a term that depends on $\{F_\sigma^{(n)}(s)\}_{n=1,2,\ldots,N}$, so "not maximizing P" is different from "not maximizing W". Define

$$A(\sigma) \quad = \quad -(r - v_0) \sum_{n=1}^{N} \alpha_n F_\sigma^{(n)}(r)$$

and recall that $W(\sigma) = P(\sigma) + A(\sigma)$ plus a constant.

As in the text, consider the case where $\sigma = pF' + (1-p)F''$ is a symmetric equilibrium. We will show that for some small (β, γ),

$$\sigma' \quad = \quad ((p+\beta)F' + (1-p-\beta)F'', (p-\gamma)F' + (1-p+\gamma)F'', pF' + (1-p)F'', \cdots, pF' + (1-p)F'')$$

gives strictly higher total surplus than σ. We will do this by finding (β, γ) such that $P(\sigma') > P(\sigma)$ and $A(\sigma') = A(\sigma)$.

By the same logic as in the text, $P(\sigma') > P(\sigma)$ as long as β and γ are both strictly positive. This is because when we start at σ, moving bidder 1 incrementally toward F' does not change P at all (since at equilibrium, bidder 1 must be indifferent among F', F'', and any mix of the two); but once bidder 1 is more likely to be playing F', bidder 2 strictly prefers F'' to F', and so increasing γ incrementally now increases P. Thus, to prove the result, we need only show that we can find $(\beta, \gamma) \gg 0$ such that $A(\sigma') \geq A(\sigma)$.

Now, since r is fixed, it must be that either $F'(r) > F''(r)$, $F'(r) = F''(r)$, or $F'(r) < F''(r)$. If $F'(r) = F''(r)$, then $A(\cdot)$ does not depend on β or γ and $A(\sigma') = A(\sigma)$, so we're done. Without loss of generality, then, focus on the csae where $F'(r) > F''(r)$. This means $A(\sigma')$ is decreasing in β and increasing in γ. So $A(\sigma') < A(\sigma)$ when $\beta > 0 = \gamma$, and $A(\sigma') > A(\sigma)$ when $\gamma > 0 = \beta$. But then by continuity, we can find β, γ both strictly positive such that $A(\sigma') = A(\sigma)$. This completes the proof.

A.3 Strict concavity of W for Theorem 5

Define

$$G(\eta_1, \ldots, \eta_K) \quad = \quad (\bar{s} - v_0) \sum_{n=1}^{N} \alpha_n - \int_r^{\bar{s}} \sum_{n=1}^{N} \alpha_n F^{(n)}(s) ds$$

as total expected surplus generated by the auction with $r = v_0$, gross of bidders' pre-auction investment costs, so that $W(\eta_1, \ldots, \eta_K) = G(\eta_1, \ldots, \eta_K) - \sum_{k=1}^{K} C(\eta_k)$. We can then write the Hessian matrix of W as $D^2 W = D^2 G - \Lambda$, where $\Lambda = \text{diag}(C''(\eta_1), C''(\eta_2), \ldots, C''(\eta_K))$. In the supplemental appendix, we calculate the second derivatives of G, to make the obvious point that if $\|\frac{\partial F_\eta}{\partial \eta}\|$ and $\|\frac{\partial^2 F_\eta}{\partial \eta^2}\|$ are all uniformly bounded, so are all the entries of $D^2 G$. So now define E such that

$$E \quad > \quad \sup_{\eta_1, \ldots, \eta_K} \sup_{z \in \mathbb{R}^K : z \cdot z = 1} z'(D^2 G) z$$

Since the entries of $D^2 G$ are all bounded, the supremum is finite. If each diagonal element of Λ is greater than E and $z \cdot z = 1$, then $z' \Lambda z > E$, so $z'(D^2 W) z = z'(D^2 G) z - z' \Lambda z < E - E = 0$ for any z with $z' z = 1$. This means $D^2 W$ is negative definite everywhere, and therefore W is strictly concave.

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Supplemental Appendix

S.1 Omitted algebra for proof of Lemma 4

As noted in the text, defining $z_i(s)$ as the probability that exactly i of the bidders other than j and k have valuations above s (and $z_{-1}(s)=0$), we can write bidder k's private gain from switching to strategy σ_k' from σ_k as

$$\begin{aligned}
U_k(\sigma_k',\cdot)-U_k(\sigma_k,\cdot) &= \int_r^{\bar{s}}\left(F_{\sigma_k}(s)-F_{\sigma_k'}(s)\right)\sum_{n=1}^{N}\alpha_n\left[F_{\sigma_j}(s)z_{n-1}(s)+(1-F_{\sigma_j}(s))z_{n-2}(s)\right]ds-c(\sigma_k')+c(\sigma_k)\\
&= \int_r^{\bar{s}}\left(F_{\sigma_k}(s)-F_{\sigma_k'}(s)\right)\sum_{n=1}^{N}\alpha_n F_{\sigma_j}(s)\left[z_{n-1}(s)-z_{n-2}(s)\right]ds\\
&\quad+\int_r^{\bar{s}}\left(F_{\sigma_k}(s)-F_{\sigma_k'}(s)\right)\sum_{n=1}^{N}\alpha_n z_{n-2}(s)ds-c(\sigma_k')+c(\sigma_k)
\end{aligned}$$

Define Z as the entire second line, and note that it does not depend on F_{σ_j}, so

$$\begin{aligned}
U_k(\sigma_k',\cdot)-U_k(\sigma_k,\cdot) &= Z+\int_r^{\bar{s}}\left(F_{\sigma_k}(s)-F_{\sigma_k'}(s)\right)\sum_{n=1}^{N}\alpha_n F_{\sigma_j}(s)\left[z_{n-1}(s)-z_{n-2}(s)\right]ds\\
&= Z+\int_r^{\bar{s}}\left(F_{\sigma_k}(s)-F_{\sigma_k'}(s)\right)F_{\sigma_j}(s)\left[\sum_{n=1}^{N}\alpha_n z_{n-1}(s)-\sum_{n=0}^{N-1}\alpha_{n+1}z_{n-1}(s)\right]ds\\
&= Z+\int_r^{\bar{s}}\left(F_{\sigma_k}(s)-F_{\sigma_k'}(s)\right)F_{\sigma_j}(s)\left[\alpha_N z_{N-1}(s)+\sum_{n=1}^{N}(\alpha_n-\alpha_{n+1})z_{n-1}(s)\right]ds
\end{aligned}$$

The rest of the proof is in the text.

S.2 Omitted algebra for proof of Lemma 6

We want to show that

$$\phi(x) \equiv \sum_{n=1}^{N}\alpha_n\sum_{i=1}^{n}\binom{K}{i-1}(1-x)^{i-1}x^{K-i+1}$$

is strictly convex. Note that

$$\begin{aligned}
\phi'(x) &= \sum_{n=1}^{N}\alpha_n\left[-\sum_{i=2}^{n}\binom{K}{i-1}(i-1)(1-x)^{i-2}x^{K-i+1}+\sum_{i=1}^{n}\binom{K}{i-1}(1-x)^{i-1}(K-i+1)x^{K-i}\right]\\
&= \sum_{n=1}^{N}\alpha_n\left[-\sum_{i=2}^{n}K\binom{K-1}{i-2}(1-x)^{i-2}x^{K-i+1}+\sum_{i=1}^{n}K\binom{K-1}{i-1}(1-x)^{i-1}x^{K-i}\right]\\
&= K\sum_{n=1}^{N}\alpha_n\left[-\sum_{i=1}^{n-1}\binom{K-1}{i-1}(1-x)^{i-1}x^{K-i}+\sum_{i=1}^{n}\binom{K-1}{i-1}(1-x)^{i-1}x^{K-i}\right]\\
&= K\sum_{n=1}^{N}\alpha_n\binom{K-1}{n-1}(1-x)^{n-1}x^{K-n}
\end{aligned}$$

and therefore

$$
\begin{aligned}
\phi''(x) &= -K\sum_{n=2}^{N}\alpha_n\binom{K-1}{n-1}(n-1)(1-x)^{n-2}x^{K-n} + K\sum_{n=1}^{N}\alpha_n\binom{K-1}{n-1}(1-x)^{n-1}(K-n)x^{K-n-1} \\
&= -K(K-1)\sum_{n=2}^{N}\alpha_n\binom{K-2}{n-2}(1-x)^{n-2}x^{K-n} + K(K-1)\sum_{n=1}^{N}\alpha_n\binom{K-2}{n-1}(1-x)^{n-1}x^{K-n-1} \\
&= -K(K-1)\sum_{n=1}^{N-1}\alpha_{n+1}\binom{K-2}{n-1}(1-x)^{n-1}x^{K-n-1} + K(K-1)\sum_{n=1}^{N}\alpha_n\binom{K-2}{n-1}(1-x)^{n-1}x^{K-n-1} \\
&= K(K-1)\sum_{n=1}^{N-1}(\alpha_n-\alpha_{n+1})\binom{K-2}{n-1}(1-x)^{n-1}x^{K-n-1} + K(K-1)\alpha_N\binom{K-2}{N-1}(1-x)^{N-1}x^{K-N-1}
\end{aligned}
$$

Since α_n is decreasing in n, this is strictly positive, so ϕ is strictly convex.

S.3 Counterexample of Inverse of Theorem 3

For an example, as per the text, suppose $N = 1$ (one prize), $K = 2$ (two bidders), $r = v_0 = 0$, and suppose that for $\eta \in [0, 1]$, F_η puts probability η on receiving valuation 1 and probability $1 - \eta$ on receiving valuation 0. Let $C(\eta) = \frac{1}{4}\eta + \frac{1}{4}\eta^2$. Note that since C is strictly convex, any mixed strategy is dominated by the pure strategy giving the same probability of valuation 1.

When the two bidders have the same valuation, both get payoff 0 from the auction, so a bidder's ex ante expected surplus is

$$U_k(\eta_k, \eta_{-k}) \quad = \quad \eta_k(1 - \eta_{-k}) - C(\eta_k)$$

and total surplus is

$$W \quad = \quad 1 - (1 - \eta_1)(1 - \eta_2) - C(\eta_1) - C(\eta_2)$$

With $C(\eta) = \frac{1}{4}\eta + \frac{1}{4}\eta^2$, the efficient symmetric level of investment maximizes $1 - (1-\eta)^2 - 2\left(\frac{1}{4}\eta + \frac{1}{4}\eta^2\right)$, which is strictly concave and maximized at $\eta = \frac{1}{2}$.

Note that Assumption 2 holds: since $F_\eta(s) = 1-\eta$ for any $s \in (0, 1)$, we have $\frac{\partial F_\eta(s)}{\partial \eta} = -1$ and $\frac{\partial^2 F_\eta(s)}{\partial \eta^2} = 0$, while $C'(\eta) = \frac{1}{4} + \frac{1}{2}\eta$ and $C''(\eta) = \frac{1}{2}$, so $0 > \frac{\frac{1}{2}}{\frac{1}{4}+\frac{1}{2}\alpha}(-1)$ holds everywhere. So $\eta_1 = \eta_2 = \frac{1}{2}$ must be an equilibrium, which we can easily verify: if $\eta_2 = \frac{1}{2}$, then $U_1(\eta_1) = \frac{1}{2}\eta_1 - \frac{1}{4}\eta_1 - \frac{1}{4}\eta_1^2$, which is concave and maximized at $\eta = \frac{1}{2}$.

However, $\eta_1 = \eta_2 = \frac{1}{2}$ does not maximize total surplus. It gives surplus $W\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{3}{4} - 2\left(\frac{3}{16}\right) = \frac{3}{8}$, but since $C(1) = \frac{1}{2}$, setting $(\eta_1, \eta_2) = (0, 1)$ or $(1, 0)$ achieves surplus of $\frac{1}{2}$. So first-best is asymmetric, but the most efficient symmetric investment level is still an equilibrium.

S.4 Omitted algebra for proof of Theorem 4

The problem

$$\max_{\eta}\left\{-\sum_{n=1}^{N}\alpha_n\int_r^{\overline{s}}\left[\sum_{i=1}^{n}\binom{K}{i-1}(1-F_\eta(s))^{i-1}(F_\eta(s))^{K-i+1}\right]ds - KC(\eta)\right\}$$

has first-order condition is

$$
\begin{aligned}
\tilde{P}'(\eta) &= -\int_r^{\bar{s}} \sum_{n=1}^{N} \alpha_n \Bigg[-\sum_{i=2}^{n} \binom{K}{i-1} (i-1)(1-F_\eta(s))^{i-2} \frac{\partial F_\eta(s)}{\partial \eta} (F_\eta(s))^{K-i+1} \\
&\qquad + \sum_{i=1}^{n} \binom{K}{i-1} (1-F_\eta(s))^{i-1} (K-i+1)(F_\eta(s))^{K-i} \frac{\partial F_\eta(s)}{\partial \eta} \Bigg] ds - KC'(\eta) \\
&= -\int_r^{\bar{s}} \sum_{n=1}^{N} \alpha_n \Bigg[-\sum_{i=2}^{n} K \binom{K-1}{i-2} (1-F_\eta(s))^{i-2} \frac{\partial F_\eta(s)}{\partial \eta} (F_\eta(s))^{K-i+1} \\
&\qquad + \sum_{i=1}^{n} K \binom{K-1}{i-1} (1-F_\eta(s))^{i-1} (F_\eta(s))^{K-i} \frac{\partial F_\eta(s)}{\partial \eta} \Bigg] ds - KC'(\eta) \\
&= -\int_r^{\bar{s}} \sum_{n=1}^{N} \alpha_n \Bigg[-\sum_{i=1}^{n-1} K \binom{K-1}{i-1} (1-F_\eta(s))^{i-1} \frac{\partial F_\eta(s)}{\partial \eta} (F_\eta(s))^{K-i} \\
&\qquad + \sum_{i=1}^{n} K \binom{K-1}{i-1} (1-F_\eta(s))^{i-1} (F_\eta(s))^{K-i} \frac{\partial F_\eta(s)}{\partial \eta} \Bigg] ds - KC'(\eta) \\
&= -K \int_r^{\bar{s}} \sum_{n=1}^{N} \alpha_n \left[\binom{K-1}{n-1} (1-F_\eta(s))^{n-1} (F_\eta(s))^{K-n} \frac{\partial F_\eta(s)}{\partial \eta} \right] ds - KC'(\eta)
\end{aligned}
$$

(The rest of the proof is in the text.)

S.5 Omitted algebra for proof of Corollary 2

$$
\begin{aligned}
\sum_{n=1}^{N} \alpha_n \binom{K-1}{n-1} (1-F_\eta(s))^{n-1} (F_\eta(s))^{K-n} &= \alpha_N \sum_{n=1}^{N} \binom{K-1}{n-1} (1-F_\eta(s))^{n-1} (F_\eta(s))^{K-n} \\
&\quad + (\alpha_{N-1} - \alpha_N) \sum_{n=1}^{N-1} \binom{K-1}{n-1} (1-F_\eta(s))^{n-1} (F_\eta(s))^{K-n} \\
&\quad + (\alpha_{N-2} - \alpha_{N-1}) \sum_{n=1}^{N-2} \binom{K-1}{n-1} (1-F_\eta(s))^{n-1} (F_\eta(s))^{K-n} \\
&\quad \cdots + (\alpha_1 - \alpha_2)(F_\eta(s))^{K-1} \\
&= \alpha_N F_\eta^{(N:K-1)}(s) + (\alpha_{N-1} - \alpha_N) F_\eta^{(N-1:K-1)}(s) \\
&\quad + \ldots + (\alpha_1 - \alpha_2) F_\eta^{(1:K-1)}(s)
\end{aligned}
$$

S.6 Omitted algebra for strict concavity of W in Theorem 5

Since

$$
G(\eta_1, \ldots, \eta_K) = (\bar{s} - v_0) \sum_{n=1}^{N} \alpha_n - \int_r^{\bar{s}} \sum_{n=1}^{N} \alpha_n F^{(n)}(s) ds
$$

where

$$F^{(n)}(s) \quad = \quad \sum_{A \in K:|A|<n} \prod_{i \in A} (1 - F_{\eta_i}(s)) \prod_{i \in K-A} F_{\eta_i}(s)$$

note that

$$\begin{aligned}
\frac{\partial G}{\partial \eta_k} \quad &= \quad -\int_r^{\bar{s}} \sum_{n=1}^{N} \alpha_n \left[\sum_{|A|<n, k \in A} \left(-\frac{\partial F_{\eta_k}}{\partial \eta_k} \right) \prod_{i \in A-\{k\}} (1 - F_{\eta_i}(s)) \prod_{i \in K-A} F_{\eta_i}(s) \right] ds \\
&\quad -\int_r^{\bar{s}} \sum_{n=1}^{N} \alpha_n \left[\sum_{|A|<n, k \notin A} \prod_{i \in A} (1 - F_{\eta_i}(s)) \left(\frac{\partial F_{\eta_k}}{\partial \eta_k} \right) \prod_{i \in K-A-\{k\}} F_{\eta_i}(s) \right] ds
\end{aligned}$$

Now, for every set A in the inner sum in the first line, the set $A - \{k\}$ shows up in the inner second line, and the summand is the same, just with the opposite sign. And for every set A in the second line with $|A| < n - 1$, the set $A \cup \{k\}$ also shows up in the first line. So the only terms that don't cancel are the sets of size exactly $n - 1$ in the second line, meaning

$$\frac{\partial G}{\partial \eta_k} \quad = \quad -\int_r^{\bar{s}} \sum_{n=1}^{N} \alpha_n \left[\left(\frac{\partial F_{\eta_k}(s)}{\partial \eta_k} \right) \sum_{A \subset K-\{k\}:|A|=n-1} \prod_{i \in A} (1 - F_{\eta_i}(s)) \prod_{i \in K-A-\{k\}} F_{\eta_i}(s) \right] ds$$

From here,

$$\frac{\partial^2 G}{\partial \eta_k^2} \quad = \quad -\int_r^{\bar{s}} \sum_{n=1}^{N} \alpha_n \left[\left(\frac{\partial^2 F_{\eta_k}(s)}{\partial \eta_k^2} \right) \sum_{A \subset K-\{k\}:|A|=n-1} \prod_{i \in A} (1 - F_{\eta_i}(s)) \prod_{i \in K-A-\{k\}} F_{\eta_i}(s) \right] ds$$

and also

$$\frac{\partial^2 G}{\partial \eta_k \partial \eta_{k'}} \quad = \quad -\int_r^{\bar{s}} \sum_{n=1}^{N} \alpha_n \left[\left(\frac{\partial F_{\eta_k}(s)}{\partial \eta_k} \right) \left(\frac{\partial F_{\eta_{k'}}(s)}{\partial \eta_{k'}} \right) \sum_{A \subset K-\{k,k'\}:|A|=n-2} \prod_{i \in A} (1 - F_{\eta_i}(s)) \prod_{i \in K-A-\{k,k'\}} F_{\eta_i}(s) \right] ds$$

The exact details here only matter to establish that if $\|\frac{\partial F_\eta}{\partial \eta}\|$ and $\|\frac{\partial^2 F_\eta}{\partial \eta^2}\|$ are uniformly bounded, so are the second derivatives of G.